How to Distribute Lots Among Farmers or is it Hard to Approximate Constrained *k*-Clustering Problem?

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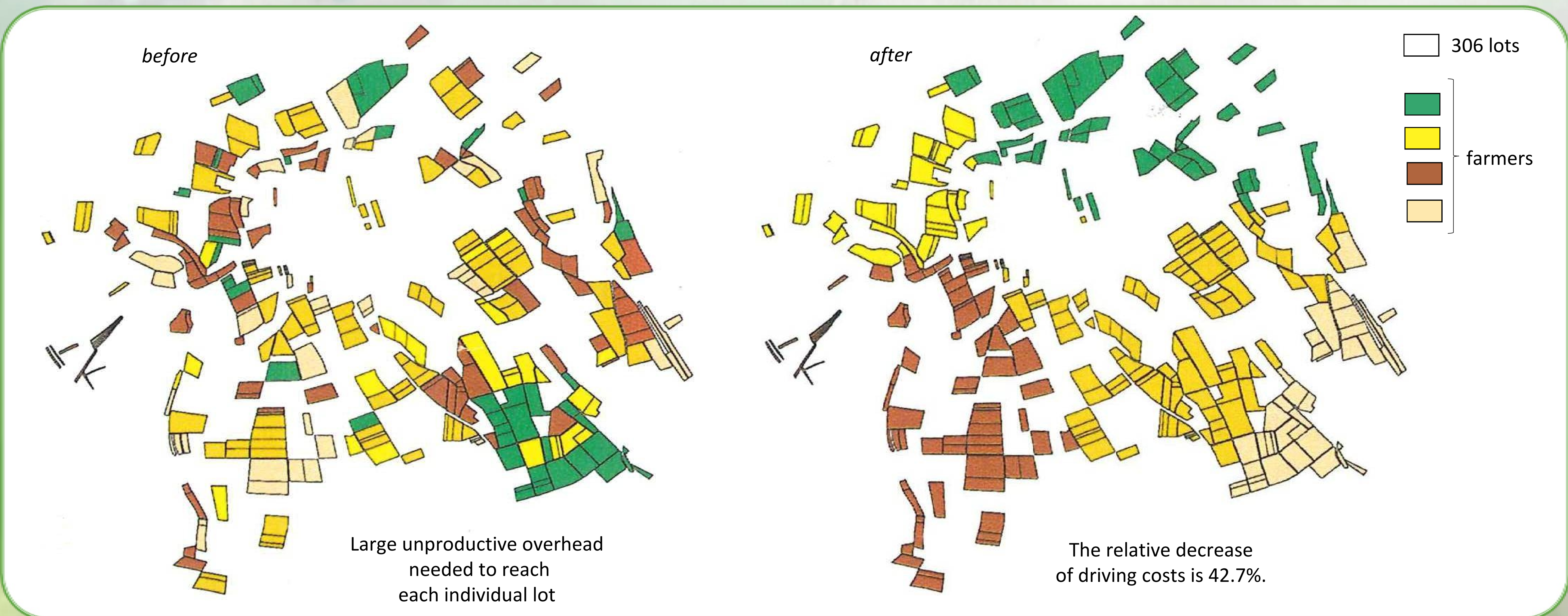

Model with known cluster centers

Formulation

Let

- $G = (V,E,w_V,w_E)$ be a complete undirected weighted graph with
- $V = \{v_1,\dots,v_m\}$
- $w_E : E \to R^+$ satisfies the triangle inequality
- $\kappa_1,\dots,\kappa_k \in (R^+)^d$ with $d \in N$
- $w_V : V \to (R^+)^d$ with $\sum_{i=1}^{k} \kappa_i = w_V(V)$
- $\mu_1,\dots,\mu_k \in N$
- $\{c_{11},\dots,c_{1\mu_1},\dots,c_{k1},\dots,c_{k\mu_k}\} \subset V$
- $\varepsilon_1^{\pm},\dots,\varepsilon_k^{\pm} \in (R^+)^d$

The goal is

- to compute a partition $C = (C_{11},\dots,C_{1\mu_1},\dots,C_{k1},\dots,C_{k\mu_k})$ of V with
- $c_{ij} \in C_{ij}$ for $i \in \{1,\dots,k\}$, $j \in \{1,\dots,\mu_i\}$
- $(1-\varepsilon_i^-) \circ \kappa_i \le \sum_{j=1}^{\mu_i} w_V(C_{ij}) \le (1+\varepsilon_i^+) \circ \kappa_i \quad (i \in \{1,\dots,k\})$

such that

$val(C) = \sum_{i=1}^{k} \sum_{j=1}^{\mu_i} \sum_{u \in C_{ij}} w_E(\{c_{ij},u\})$ is minimal among all such partitions

Method

ILP-model → LP-relaxation with $\varepsilon_i^{\pm} = 0$ → Optimum → Rounding fractional variables → Feasible solution

- ILP-model: ◦ mk variables ◦ $2k+m$ constrains
- Optimum: at most k fractional assigned points
- Feasible solution: on all tested instances $\varepsilon_i^{\pm} = \pm 3\%$

For *7* farmers and *651* lots according bonity and size
– decreasing till *10,7%* cultivation
– and *50,42%* driving costs

Model with unknown cluster centers

Formulation

Let

- $k, d \in N$
- set $X = \{x_1,\dots,x_m\}$ of m points of R^d
- non-negative integral k-vector $(\kappa_1,\dots,\kappa_k)^T$
- $w_V : X \to (R^+)^d$ with $\sum_{i=1}^{k} \kappa_i = w_V(X)$
- $\varepsilon_1^{\pm},\dots,\varepsilon_k^{\pm} \in (R^+)^d$
- Σ denote the set of feasible clusterings of X whereby

Feasible clustering C of X consists of

- k pairwise disjoint subsets $C_1,\dots,C_k$ of X such that
- $(1-\varepsilon_i^-) \circ \kappa_i \le w_V(C_i) \le (1+\varepsilon_i^+) \circ \kappa_i,\ 1 \le i \le k$
- $\bigcup_{i=1}^{k} C_i = X.$

For any given cluster C_i

- c_i be its *center of gravity*, $c_i = 1/\kappa_i \sum_{j=1}^{\kappa_i} w_V(x_{ij}) x_{ij}$, where
- $x_{i1},\dots,x_{i\kappa_i}$ are the points contained in C_i.

The goal is

- to compute a clustering $C^* \in \Sigma$ of X such that
- C^* is maximal among all $C \in \Sigma$ according to
- $val(C) = \sum_{i=1}^{k} \sum_{j=i+1}^{k} \| c_i - c_j \|_p^p$

Method

QIP-model → QP-model → The polytope P described by the constraints → Approximation of the objective function by the polytope $K = \bigcap_{i=1}^{h} \{y : f_i^T y \le 1\}$, $B_2^{dk(k-1)/2} \subset K \subset (1+\varepsilon) B_2^{dk(k-1)/2}$ → Feasible solution

- QIP-model: ◦ *km+dk(k-1)/2* variables ◦ *dk(k-1)/2* of them in objective function
- QP-model: ◦ LP-relaxation with $\varepsilon_i^{\pm} = 0$ ◦ objective is a norm of *dk(k-1)/2* variables ◦ linear constraints
- The polytope P: dimension *km+dk(k-1)/2*
- Feasible solution: choose the argument of $\max_{1 \le i \le h} \max_{y \in P} f_i^T y$ and compute a corresponding integral solution

For $w_V(x_j) = 1,\ 1 \le j \le m$, PTAS thats running time is of order $\varepsilon^{-dk(k-1)}$

For *13* farmers and *861* lots according bonity and size
– decreasing till *57,9%* driving costs

S.Borgwardt, A. Brieden, P.Gritzmann: Constrained Minimum k-star Clustering and its Application to the Consolidation of Farmland, Operation Research, 2009

A.Brieden, P.Gritzmann: A quadratic optimization model for the consolidation of farmland by means of lend-lease agreements, Operation Research Proceedings 2003, Selected Papers of the International Conference on Operation Research (OR2003), Springer-Verlag, Heidelberg, 2003, 324-331.